Handling free variables in primal-dual interior-point methods using a quadratic cone

Erling D. Andersen
MOSEK ApS
C/O Symbion Science Park
Fruebjergvej 3, Box 16
DK 2100 Copenhagen O
www.mosek.com
Email: e.d.andersen@mosek.com
WWW: erling.andersen.name

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Outline

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Introduction

Problem:

$$
\begin{array}{llll}
(P) & \text{minimize} & c^T x + \bar{c}^T \bar{x} & \\
 & \text{subject to} & Ax + \bar{A}\bar{x} & = \; b, \\
 & & x \geq 0. &
\end{array}
$$

where $x \in R^n$ and $\bar{x} \in R^{\bar{n}}$.

Notes

- Has free variables.

Solution methods:

- Simplex based methods.
  - Free variables are good.

- **Primal-dual based interior-point methods.**
  - Efficient both in theory and practice.
  - Free variables are bad.

Main work in primal-dual IPMs:

$$\begin{bmatrix} -D & & A^T \\ & 0 & \bar{A}^T \\ A & \bar{A} & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_{\bar{x}} \\ d_y \end{bmatrix} = \begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix}$$

where D is a positive definite diagonal matrix.

Solution methods:

- No free variables.

$$\begin{aligned} ADA^T d_y &= r^2 + AD^{-1} r^1, \\ d_x &= -D^{-1}(r^1 - A^T d_y). \end{aligned}$$

- Free variables:
  - Symmetric in-definite factorization.
  - Suggested by Mehrotra and others.
  - Computationally expensive.

Alternative methods:

- Explicit elimination of free variables (RTV book).
  - Can create fill-in and bad scaling.
- Split the free variables
$$\bar{x} = \bar{x}^+ - \bar{x}^-, \quad \bar{x}^+, \bar{x}^- \geq 0.$$
  - Popular method.
  - Create an unbounded solution set.
  - Numerical unstable.
  - Required ad-hoc tricks.
  - Complexity increases with $2\bar{n}$.
- Perturbation approach.
  - Cs. Meszaros.
  - Somewhat ad-hoc.

Summary:

- Several methods for handling free variables.
- All have drawbacks.
- Ready for a new method.

Theory

Dual problem:

$$
(D) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & \begin{array}{rcl} A^T y + s & = & c, \\ \bar{A}^T y & = & \bar{c}, \\ s \geq 0. & & \end{array} \end{array}
$$

Lemma:

a. (P) has an optimal solution if and only if there exist $(x^*, \bar{x}^*, y^*, s^*)$ such that

$$\begin{array}{rcl} Ax^* + \bar{A}\bar{x}^* & = & b, \\ A^T y^* + s^* & = & c, \\ \bar{A}^T y & = & \bar{c}, \\ x^*, s^* & \geq & 0 \end{array}$$

and

$$c^T x^* + \bar{c}^T \bar{x}^* - b^T y^* = (x^*)^T s^* = 0.$$

b. (P) is infeasible if and only if there exist y^* such that

$$A^T y^* \leq 0, \;\; \bar{A}^T y^* = 0, \;\; b^T y^* > 0.$$

c. (D) is infeasible if and only if there exist x^* such that

$$Ax^* + \bar{A}\bar{x}^* = 0, \;\; c^T x^* + \bar{c}^T \bar{x}^* < 0, \;\; x^* \geq 0.$$

New approach

Quadratic cone approach:

$$
\begin{array}{llll}
(QCF) & \text{minimize} & c^T x + \bar{c}^T \bar{x} & \\
 & \text{subject to} & Ax + \bar{A}\bar{x} & = \ b, \\
 & & \|\bar{x}\| & \leq \ \bar{x}', \\
 & & x, \bar{x}' \geq 0, &
\end{array}
$$

where $\bar{x}' \in R$

Rotated quadratic cone approach:

$$
\begin{array}{llll}
(RQCF) & \text{minimize} & c^T x + \bar{c}^T \bar{x} & \\
 & \text{subject to} & Ax + \bar{A}\bar{x} & = \ b, \\
 & & \bar{x}'_1 & = \ 1, \\
 & & \|\bar{x}\|^2 & \leq \ 2\bar{x}'_1 \bar{x}'_2, \\
 & & x, \bar{x}' \geq 0, &
\end{array}
$$

where $\bar{x}' \in R^2$.

- $\bar{x}'$ is a new variable.
- $\|\cdot\|$ is the Euclidean norm.
- New constraints are **redundant**.

Facts:

- Solved efficiently using NT primal-dual alg.

- Iteration complexity.
  - Elimination: $O(\sqrt{n}\varepsilon^{-1})$.
  - Split free: $O(\sqrt{(n+2\bar{n})}\varepsilon^{-1})$.
  - New approach: $O(\sqrt{(n+2)}\varepsilon^{-1})$

<u>Duality?</u>

Questions:

- What are the dual problems?

- Bad duality states?

Dual problem to (QCF):

$$
\begin{array}{llrcl}
(DQCF) & \text{maximize} & b^T y & & \\
 & \text{subject to} & A^T y + s & = & c, \\
 & & \bar{A}^T y + \bar{s} & = & \bar{c}, \\
 & & \bar{s}'_1 & = & 0, \\
 & & \|\bar{s}\| & \leq & \bar{s}'_1, \\
 & & s, \bar{s}' \geq 0. & &
\end{array}
$$

Dual problem to $(RQCF)$:

$$
\begin{array}{llrcl}
(DRQCF) & \text{maximize} & b^T y + \tilde{y} & & \\
 & \text{subject to} & A^T y + s & = & c, \\
 & & \bar{A}^T y + \bar{s} & = & \bar{c}, \\
 & & \tilde{y} + \bar{s}'_1 & = & 0, \\
 & & \bar{s}'_2 & = & 0, \\
 & & \|\bar{s}\|^2 & \leq & 2\bar{s}'_1 \bar{s}'_2, \\
 & & s, \bar{s}' \geq 0. & &
\end{array}
$$

Observations:

- Primal has unbounded solution set.
- Interior of dual solution set is empty.
- Potential unstable (as splitting approach).

Lemma:

Assume (P) has an optimal solution then:

a.) Both the primal problem (QCF) and the corresponding dual problem $(DQCF)$ has an optimal solution and the duality gap between them is zero.

b.) Both the primal problem $(RQCF)$ and the corresponding dual problem $(DRQCF)$ has an optimal solution and the duality gap between them is zero.

Cases:

- (P) has an optimal solution: **Good**.
- What if (P) is infeasible?

Lemma: (QCF)

a.) (QCF) is infeasible if and only if (P) is infeasible. Furthermore, (QCF) is infeasible if and only if there exist $(y^*, \bar{s}^*, \bar{s}'^*)$ such that

$$\begin{array}{rcl} A^T y^* = -s^* & \leq & 0, \\ \bar{A}^T y^* = -\bar{s}^* & = & 0, \\ \bar{s}'^* = 0, \ b^T y^* & > & 0. \end{array}$$

b.) (QCF) is infeasible if and only if (D) is infeasible. Furthermore, (QCF) is infeasible if and only if there exist $(x^*, \bar{x}^*, \bar{x}'^*_1)$ such that

$$\begin{array}{rcl} x^* & \geq & 0, \\ \bar{x}'^*_1 & \geq & \|\bar{x}^*\|, \\ Ax^* + \bar{A}\bar{x}^* & = & 0, \\ c^T x^* + \bar{c}^T \bar{x}^* & < & 0. \end{array}$$

- Good situation for (QCF)

- Essentially (P) and (QCF) share certificates of infeasibility.

Lemma: $(RQCF)$

a.) $(RQCF)$ is infeasible if and only if (P) is infeasible. Furthermore, $(RQCF)$ is infeasible if and only if there exist $(y^*, \tilde{y}^*, \bar{s}^*, \bar{s}'^*)$ such that

$$\begin{array}{rcl} A^T y^* = -s^* & \leq & 0, \\ \bar{A}^T y^* = -\bar{s}^* & = & 0, \\ -\tilde{y}^* = \bar{s}'^* & \geq & 0, \\ b^T y^* + \tilde{y}^* & > & 0. \end{array}$$

b.) $(RQCF)$ is infeasible if and only if (D) is infeasible. Furthermore, $(RQCF)$ is infeasible if and only if there exist $(x^*, \bar{x}^*, \bar{x}'^*)$ and an $\epsilon \in (0, 1)$ such that

$$\begin{array}{rcl} x^* \geq 0,\ \bar{x}'^*_1 & = & \epsilon^k, \\ \bar{x}'^*_2 & \geq & \frac{\|\bar{x}^*\|}{\epsilon^k}, \\ Ax^* + A\bar{x}^* & = & 0, \\ c^T x^* + \bar{c}^T \bar{x}^* & < & 0. \end{array}$$

for $k \to \infty$.

- Half-good situation for (QCF)

Example:

$$\text{minimize} \quad x_1.$$

Primal:

$$\begin{array}{lll} \text{minimize} & x_1 & \\ \text{subject to} & x_2 & = 1, \\ & x_1^2 & \le 2x_2x_3, \\ & x_2, x_3 \ge 0. & \end{array}$$

"Certificate:"

$$x_1 = -1, \quad x_2 = \varepsilon, \quad x_3 = \frac{1}{2}\varepsilon^{-1}.$$

Algorithmic issues

- Primal-dual algorithm.
- Nesterov-Todd Search direction.
- Usual stuff possible:
  - Mehrotra predictor-corrector.
  - Gondzio correctors.
  - Different primal and dual stepsize.
- Linear algebra is slightly more expensive.
- Linear system?

The case of problem (QCF)

Must solve a system of the form:

$$(AXS^{-1}A^T + \theta^2 \bar{A}\bar{A}^T + vv')h = r$$

Comments

- Usual system plus rank-1 term.
- Easy.
- Slightly more costly computationally.

The case of problem $(RQCF)$

Must solve a system of the form:

$$(AXS^{-1}A^T + \theta^2 \bar{A}\bar{A})h = r$$

Summary:

- Usual system.
- No rank-1 term.

Computational results

- Software (**http://www.mosek.com**):
  - MOSEK 2.1RC.
    - ∗ C/MATLAB/AMPL/GAMS/MPS interfaces.
- Computer:
  - 1.13GHZ P3, 512MB RAM. WXP.
- Method.
  - (QCF) method only.

Problems:

Name	Before presolve		After presolve	
	Con-straints	Varia-bles	Con-straints	Varia-bles.
agg3	516	302	287	267
cycle	1903	2857	830	1937
perold	625	1312	471	1152
pilot-ja	940	1677	669	1459
pilot-we	722	2711	547	2379
pilot4	410	970	308	812
stocfor3	16675	15695	6692	8772
1540a	1740	1275	645	593
abhillsf	578	447	498	351
cinema92	4905	12980	2257	10332
kelen10	7728	4100	4716	3661
kelen5	8669	4100	5316	3661
nrc	3539	6324	1519	4295
os10	7728	4100	4716	3661
os5	8669	4100	5316	3661
os6	25591	11066	15855	10119
osg39	25414	11564	15876	9924
osg41	26168	12292	15712	10064
pole	3031	2185	2073	1983
racv	2384	697	2025	686
ste	4013	5110	1430	2772
baxter	27441	15128	18436	10787
complex	1023	1398	1011	1386
pf2177	9728	630	9662	630
df2177	630	9728	594	9573
rlfddd	4050	57471	4048	40343
rlfdual	8052	66918	4044	40338
rlfprim	58866	8052	36294	4046
Sum	261738	265189	161847	189644

Name	Con-straints	Varia-bles	Free var.
agg3	257	551	47
cycle	830	2183	5
perold	471	1243	71
pilot-ja	669	1645	71
pilot-we	547	2477	56
pilot4	308	903	76
stocfor3	6692	13558	0
1540a	588	801	509
abhillsf	351	870	100
cinema92	2257	10332	624
kelen10	3231	7947	56
kelen5	3231	8547	56
nrc	1519	5594	29
os10	3231	7947	56
os5	3231	8547	56
os6	8829	24684	142
osg39	9854	26261	371
osg41	9990	26233	507
pole	2029	4551	71
racv	991	4000	328
ste	1430	4018	17
baxter	10765	29213	4593
complex	1011	1386	373
pf2177	630	10922	384
df2177	594	10167	414
rlfddd	4048	44391	98
rlfdual	4044	44382	98
rlfprim	4046	40340	100
Sum	85674	343693	9308

Accuracy and efficiency:

Name	Primal obj.	Sig. fig.	Iter.	Simplex iter.	Time
agg3	1.0312e+007	8	17	0	0.1
cycle	-5.2264e+000	8	24	0	0.5
perold	-9.3808e+003	10	33	0	0.3
pilot-ja	-6.1131e+003	10	38	2	0.8
pilot-we	-2.7201e+006	11	26	0	0.3
pilot4	-2.5811e+003	10	24	1	0.1
stocfor3	-3.9977e+004	10	31	0	3.6
1540a	-1.0479e+001	8	17	0	2.4
abhillsf	5.5138e+006	9	23	0	0.1
cinema92	1.3968e+000	7	37	0	4.5
kelen10	1.1369e+008	9	30	0	3.7
kelen5	9.5578e+007	10	19	0	2.4
nrc	-3.6678e+004	6	33	1	2.7
os10	1.1369e+008	9	30	0	3.8
os5	9.5578e+007	10	19	0	2.4
os6	9.6153e+007	10	28	0	9.2
osg39	2.7041e+005	6	30	0	7.9
osg41	2.7041e+005	8	25	5	5.3
pole	2.4690e+004	7	64	0	2.0
racv	1.8287e-012	9	8	0	0.2
ste	9.5407e+002	12	30	0	1.3
baxter	5.6007e+007	8	37	1	13.3
complex	-9.9667e+001	10	15	0	10.0
pf2177	9.0000e+001	10	8	0	0.8
df2177	9.0000e+001	9	6	0	0.6
rlfddd	-1.3000e+001	7	8	0	6.6
rlfdual	-1.0000e+000	8	9	0	7.7
rlfprim	1.0000e+000	5	9	0	7.3

Conclusion

- Promising method.

- No heuristics.

- Future work:
  - Improved efficiency.
  - Try detecting more free variables.